Subspaces:

The definition: W is a subspace of V if it is simultaneously a vector space with the same addition and multiplication as V and if its vectors are a subset of V's vectors.

So, a subspace uses the same addition and multiplication, and simply fits inside another vector space. They are typically written $W \subseteq V$. Every vector space V has the following two subspaces:

$$\{\mathbf{0}\} \subseteq V, \qquad V \subseteq V.$$

Those are the 'trivial' subspaces. Every vector space has its own version of those. Anything else is classified as a 'proper' subspace.

If $W \subseteq V$ and $V \subseteq W$ then $V = W$.

Example: First: you *Can Not* describe $\mathbb{R}^2$ as a subspace of $\mathbb{R}^3$, since their vectors are completely incompatible, as are their addition and multiplication. The following, however, is true:

$$\left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}, x, y \in \mathbb{R} \right\} \subseteq \mathbb{R}^3,$$

which is a very similar thing, really.

Example: $\mathcal{P}_2 \subseteq \mathcal{P}_3$. Notice yet another difference between the $\mathbb{R}^n$ spaces and $\mathcal{P}_n$ spaces.

Example: The space of polynomials $\mathcal{P}$ is a subspace of the space of functions $\mathcal{F}$.

Example: The plane $x - 2y + 3z = 2$ is NOT a subspace of $\mathbb{R}^3$. We can confirm this with at simple example. The vector $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ is on the plane. However, $\mathbf{v} + \mathbf{v} = 2\mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$ is NOT, as it does not satisfy the equation. This violates the FIRST AXIOM of the addition. The closure axiom. This is the normal way to check.

All possible subspace candidates need to use the SAME ADDITION AND MULTIPLICATION. As a result, most of the axioms need not be checked, since the two operations are already known to work.

The Subspace Test:

There are three things that must be checked to confirm that a set of vectors W within V is a subspace. Just three.

Theorem: Assume that all vectors in W are in V. Then, W is a subspace of V if W meets these three properties:

1. The zero vector ($\mathbf{0} \in V$) is in W.
2. W is closed under vector addition.
3. W is closed under scalar multiplication.

proof:

Assume W has all three properties. We will now prove that W must then have ALL the necessary properties of a vector space.

The first property means that W is non-empty, an important requirement. Now to go through the ten axioms.

A1 is simply property 2. A2 and A3 are true as the addition is the same as for V. A4 is simply property 1. A5 is done via property 3: simply use $-1 \times \mathbf{x}$ (or $(-1) \boxtimes \mathbf{x}$, if you prefer). This is not done via an axiom, but with one of the derived properties of vector spaces (we called this one B4).

S1 is property 3. S2-S5 are all covered by the properties of V itself. That's all of them, the proof is now finished.

That's it. Now, some may notice that the first requirement could be more general, simply a 'W is non-empty.' Combining 'non-empty' with the third property gives you $\mathbf{0}$, since if you have $\mathbf{x}$ you can get $\mathbf{0} = 0 \boxtimes \mathbf{x}$. This is true, but the zero test *should* be there for a few reasons.

- Checking for $\mathbf{0}$ is usually much easier than for any other vector, and we need to find one.
- Finally, not having $\mathbf{0}$ in W is very common in cases where W is not a subspace, so it is a good strategy to check that first.

so, it's easier this way.

Example: Homogeneous equations are of the form $a_n x_n + a_{n-1} x_{n-1} + \cdots + a_1 x + a_0 = 0$, a zero constant term. Lets check if the set of those in $\mathcal{E}_3$ forms a subspace.

The zero element in $\mathcal{E}_3$ is simply $0x_3 + 0x_2 + 0x_1 = 0$. This is a homogeneous equation.

Adding:

$$a_3x_3+a_2x_2+a_1x_1 = 0 \quad \boxplus \quad b_3x_3+b_2x_2+b_1x_1 = 0 \quad \Longrightarrow \quad (a_3+b_3)x_3+(a_2+b_2)x_2+(a_1+b_1)x_1 = 0$$

so yes.

Scalar Multiplication:

$$d \boxtimes [a_3x_3 + a_2x_2 + a_1x_1 = 0] \quad \Longrightarrow \quad (da_3)x_3 + (da_2)x_2 + (da_1)x_1 = 0$$

and we are done.

Example: Is $V = \{f(x)|f(1) = 0\}$ a subspace of $\mathcal{F}_{(0,2)}$?

We just need to check the three parts of the subspace test. First, is the zero function in the set? Yes, since it would be zero at $x = 1$. Now to check addition.

Take $g(x)$ such that $g(1) = 0$ and $h(x)$ such that $h(1) = 0$.

$$(g + h)(x) = g(x) + h(x), \qquad (g + h)(1) = g(1) + h(1) = 0$$

so V is closed under addition.

Now, multiplication, take $g(x)$ in V and $a \in \mathbb{R}$.

$$ag(1) = a(0) = 0$$

so yes, it is a subspace.

Spans

Definition: the Span of a (non-empty) set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_2, \dots, \mathbf{v}_n\} \in V$ is the set of all linear combinations of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. So:

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n, \text{ for all } a_1, a_2, \dots, a_n \in \mathbb{R}\}.$$

Again, the set of ALL linear combinations. In essence, everything you can make out of the vectors v_1 to v_n, using vector addition and scalar multiplication.

Lets take a look at a few properties of spans.

First: the zero vector:

$$\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n,$$

which is a perfectly reasonable linear combination.

Next: addition. Take two elements

$$\mathbf{a} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

and

$$\mathbf{b} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n.$$

Add them together for

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) + (b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n) \\ &= (a_1 + b_1)\,\mathbf{v}_1 + (a_2 + b_2)\,\mathbf{v}_2 + \cdots + (a_n + b_n)\,v_n,\end{aligned}$$

another linear combination.

Next we check multiplication

$$\begin{aligned}c\,\mathbf{a} &= c\,(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) \\ &= (ca_1)\mathbf{v}_1 + (ca_2)\mathbf{v}_2 + \cdots + (ca_n)\mathbf{v}_n,\end{aligned}$$

yet another linear combination.

So: we can say with certainty that

Theorem: The Span$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, all $\mathbf{v}_k \in V$, is a Subspace of V.

This is saying something else, too. Namely:

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ all belong to the subspace $W \subseteq V$ then

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

is a subspace of W. This is actually sort of saying the same thing, since W is a vector space itself.

We can expand this even further.

Corollary: $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is the smallest possible subspace containing $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$.

How do we get this? It's actually quite easy.
proof:
Assume that $W \subseteq \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and that all $\mathbf{v}$ vectors belong to W. Due to the earlier Theorem, that means $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq W$, which combines with the other $\subseteq$ to give us $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = W$.

So, the span of a set of vectors in V is always the smallest subspace of V containing that set of vectors. Basically, this is an alternate definition of a span. It also fits in nicely with one description of spans: they are everything you can make out of a set of vectors using the vector space operations. Think about it.

Example:
Lets look at the span of the following set

$$\left\{x^2 + x, \quad x^2 + 1, \quad x - 1, \quad 1\right\} \in \mathcal{P}_2.$$

The elements of the span will be of the form

$$\begin{aligned} &b_1(x^2 + x) + b_2(x^2 + 1) + b_3(x - 1) + b_4(1) \\ =&(b_1 + b_2)x^2 + (b_1 + b_3)x + (b_2 - b_3 + b_4) \end{aligned}$$

Lets see which elements of $\mathcal{E}_3$ we can match with this:

$$a_2x^2 + a_1x + a_0 = (b_1 + b_2)x^2 + (b_1 + b_3)x + (b_2 - b_3 + b_4)$$

so

$$b_1 + b_2 = a_2 \qquad b_1 + b_3 = a_1 \qquad b_2 - b_3 + b_4 = a_0.$$

Four variables, three equations.

First, $b_1 = a_2 - b_2$. This makes

$$a_2 - b_2 + b_3 = a_1 \quad \Longrightarrow quad b_3 = a_1 - a_2 + b_2.$$

Finally,

$$b_4 + b_2 - (a_1 - a_2 + b_2) = a_0 \quad \Longrightarrow \quad b_4 = a_0 + a_1 - a_2 - b_2.$$

We've satisfied every equation, and here's the result:

$$b_1 = a_2 - b_2 \qquad b_3 = a_1 - a_2 + b_2 \qquad b_4 = a_0 + a_1 - a_2 - b_2$$

with b_2 arbitrary. No matter what polynomial we want in $\mathcal{P}_2$, we can make it out of those vectors. That set of vectors actually spans the whole space. We have a term for that.

Spanning Sets:

We will spend some time finding these. A spanning set is just what the name suggests. You have a vector space V, and a set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ is a spanning set if

$$V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}.$$

Note: a spanning set can have redundant terms, or even zero, in its set. Efficiency is NOT part of the definition. It just has to be *inside* the space and its span should cover the entire space.

The previous example spanned $\mathcal{P}_2$, but the simpler one is

$$\left\{1,\, x,\, x^2,\right\},$$

however, this works too:

$$\left\{1, x, x^2, 0, 12, -x^2+1, x^2+2\right\}.$$

Again, it doesn't have to be efficient.

Here's another basic example:

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}.$$

Things to pay attention to:

- Spanning sets are for vector spaces, subspaces, etc. Spans create vector spaces, so spanning sets only really apply to vector spaces.
- If you can totally equate something with a span, then it is a subspace. So, for instance, any plane in $\mathbb{R}^3$ of the form

  $$\mathbf{x} = \mathbf{b}t + \mathbf{c}s \qquad t, s \in \mathbb{R}$$

  is a subspace, since that is a span, etc...

  If something is NOT written like a span, this is no help, for instance

  $$\mathbf{x} = \mathbf{a} + \mathbf{b}t + \mathbf{c}, \qquad t, s \in \mathbb{R}, \qquad \mathbf{a} \neq \mathbf{0}$$

  is NOT a subspace.
- Spanning sets allow you to characterize an entire vector space using only a small set of vectors. This is helpful, frequently.
- Remember: spanning sets need to be INSIDE the space in question and span it. NOTHING ELSE IS REQUIRED. They can have unnecessary terms.

Now a few exercises from the text, Section 5.1

6.b)d)f). deg(p(x)) = the highest power of x in the polynomial

7.b)

8.b)d)f).

9.b)d). remember how spans relate to linear combinations (remember those too)